Exotic non-Abelian anyons from conventional fractional quantum Hall states: Supplementary Information

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Supplementary Methods

Parafermion Josephson effect

In the main text we deduced the qualitative dependence of the energy, and hence the Josephson current, on the phase difference $\delta\phi_{\rm sc}$ between the two superconductors in Fig. 2(a). This Section explores the physics uncovered there more quantitatively. We continue to work in the limit where the tunnelling strength $\mathcal{M}(x)$ vanishes whereas the pairing fields in Eq. (15) pin θ beneath each superconductor. The normal region between the superconductors can then be described by an effective Hamiltonian

$$H = \frac{mv}{2\pi}\int_{x_1}^{x_2+\ell} dx[(\partial_x\varphi)^2 + (\partial_x\theta)^2], \qquad \text{(S1)}$$

subject to boundary conditions on $\varphi(x_1)$ and $\varphi(x_2+\ell)$ induced by the neighboring superconductors.

Because (in contrast to the earlier discussion of the solution for localized zero-modes in Methods section) the same field is now pinned at both endpoints, it is essential that one incorporates compactness of φ in what follows; failure to do so yields incorrect results for the dependence of the energy on $\delta\phi_{\rm sc}$. We will for simplicity set $\varphi(x_1) = 0$—that is, we fix the eigenvalue of the operator $\hat{n}_\varphi^{(1)}$ defined earlier to zero without loss of generality. At the right boundary, however, we take

$$\varphi(x_2+\ell) = \text{mod}\left[\frac{\pi}{m}\left(\hat{n}_\varphi^{(2)} + \frac{\delta\phi_{\rm sc}}{2\pi}\right) + \pi, 2\pi\right] - \pi, \qquad \text{(S2)}$$

where $\hat{n}_\varphi^{(2)}$ is the same integer-valued operator introduced previously. The right-hand side of Eq. (S2) minimizes the pairing term in Eq. (15) and importantly also restricts $\varphi(x_2+\ell)$ to lie between $-\pi$ and π for any $\delta\phi_{\rm sc}$ and $\hat{n}_\varphi^{(2)}$. Imposing this bound on the range of $\varphi(x_2+\ell)$ ensures that $\varphi(x)$ need not exhibit any unnecessary twists between $x = x_1$ and $x_2+\ell$.

To diagonalize the Hamiltonian we decompose φ, θ as follows:

$$\begin{aligned} \varphi(x) &= \varphi(x_2+\ell)\,\frac{x-x_1}{x_2+\ell-x_1} \\ &+ \frac{1}{\sqrt{m}}\sum_{k=1}^{\infty}\frac{\sin\tilde{\lambda}_k(x)}{\sqrt{k}}i\left(a_k - a_k^\dagger\right) \qquad \text{(S3)} \\ \theta(x) &= \theta_0 + \frac{1}{\sqrt{m}}\sum_{k=1}^{\infty}\frac{\cos\tilde{\lambda}_k(x)}{\sqrt{k}}\left(a_k + a_k^\dagger\right), \qquad \text{(S4)} \end{aligned}$$

where $\tilde{\lambda}_k(x) = \frac{k\pi(x-x_1)}{x_2+\ell-x_1}$ and as usual a_k are canonical bosons satisfying $\left[a_k, a_{k'}^\dagger\right] = \delta_{k,k'}$. In Eq. (S4) θ_0 represents the zero-momentum component of $\theta(x)$ (note that $k=0$ is excluded from both sums above). The boundary conditions on $\varphi(x)$ are clearly obeyed in this representation, while the commutation relations between φ, θ in Eq. (6) are also preserved provided θ_0 and $\hat{n}_\varphi^{(2)}$ are conjugate variables satisfying $\left[\hat{n}_\varphi^{(2)}, \theta_0\right] = \mathrm{i}$. Using the decomposition in Eqs. (S3) and (S4) one can express the effective Hamiltonian as

$$\begin{aligned} H &= \sum_{k=1}^{\infty}\tilde{\epsilon}_k\left(a_k^\dagger a_k + 1/2\right) + \mathcal{E}(\delta\phi_{\rm sc}) \qquad \text{(S5)} \\ \mathcal{E}(\delta\phi_{\rm sc}) &= \frac{mv}{2\pi}\frac{[\varphi(x_2+\ell)]^2}{x_2+\ell-x_1}, \qquad \text{(S6)} \end{aligned}$$

with $\varphi(x_2+\ell)$ given by Eq. (S2). The first term in H above simply describes gapped excitations with energy $\tilde{\epsilon}_k = \frac{\pi v}{x_2+\ell-x_1}k$, which we assume are absent. More interestingly, the second term captures the dependence of the energy on the superconducting phase difference imposed across the junction.

Since $[H, \hat{n}_\varphi^{(2)}] = 0$ the eigenvalue of $\hat{n}_\varphi^{(2)}$ is a conserved quantity that can not change under adiabatic evolution of the Hamiltonian. It is this crucial property that gives rise to fractional Josephson effects. For a fixed initial value of $\hat{n}_\varphi^{(2)}$, one sees from Eqs. (2) and (6) that the energy is $4\pi m$ periodic in $\delta\phi_{\rm sc}$, despite the fact that the underlying Hamiltonian—recall Eq. (15)—clearly exhibits 2π periodicity. [Note that here is where compactness of φ is essential. Had we expressed $\varphi(x_2+\ell)$ in Eq. (S2) without modding by 2π, the energy would increase unboundedly with $\delta\phi_{\rm sc}$, which is obviously physically incorrect.] As a concrete illustration, Supplementary Figure S1 displays the energy $\mathcal{E}(\delta\phi_{\rm sc})$ versus $\delta\phi_{\rm sc}$ for the six inequivalent $\hat{n}_\varphi^{(2)}$ values in the $m=3$ case.

As mentioned in the main text the Josephson current flowing across the junction exhibits the same $4\pi m$ periodicity as the energy. One should, however, bear in mind the following caveats that have been raised in the context of the Majorana-mediated fractional Josephson effect (see, e.g., Refs. 8, 32). In any experiment the measured current will consist of a $4\pi m$-periodic contribution arising from the fused parafermions and a conventional 2π-periodic component flowing in parallel. (The latter can arise, for example, from the ordinary Josephson current that flows directly between the two parent s-wave superconductors.) These currents must be disentangled if one

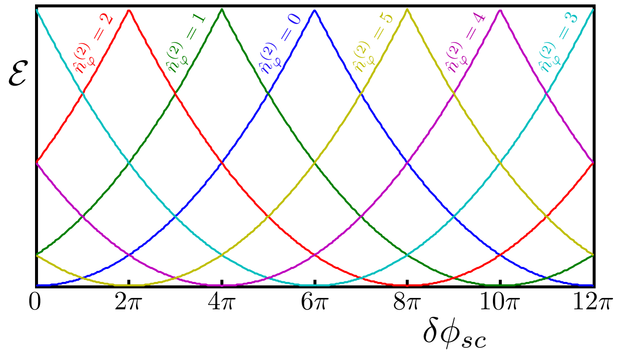

FIG. S1: **Parafermion Josephson effect**. Energy versus superconducting phase difference $\delta\phi_{\rm sc}$ across the Josephson junction in the $m = 3$ case. The six curves shown correspond to the distinct values of $\hat{n}_\varphi^{(2)}$ characterizing the pinning of φ under the right superconductor, assuming that $\varphi = 0$ beneath the left superconductor. Provided $\hat{n}_\varphi^{(2)}$ is conserved the energy and hence the current are both 12π periodic in $\delta\phi_{\rm sc}$.

is to utilize Josephson measurements to read out the qubits encoded by the parafermions. We also note that in practice various imperfections—e.g., inelastic processes that change the value of $\hat{n}_\varphi^{(2)}$ or additional parafermion couplings that spoil conservation of $\hat{n}_\varphi^{(2)}$—can potentially restore 2π periodicity of the current. Exploring these subtleties in detail would be quite interesting, particularly given the fractionalized nature of the system we are dealing with.

Quasiparticle tunnelling at the constriction in Fig. 3(b)

The Parafermion Braiding section of the main text noted that at the constriction in Fig. 3(b) charge e/m excitations can tunnel between the right-moving (red) edge states, whereas only electrons can tunnel between the left-moving (blue) edge states. The distinction between these allowed processes is crucial for the outcome of the braid analyzed there. We now wish to elaborate on this point by examining the tunnelling Hamiltonian in greater detail. Let $-x_0$ and $+x_0$ respectively denote the coordinates of the left and right sides of the constriction where tunnelling takes place. One can then model the coupling of right- and left-moving modes across the sack by

$$\begin{aligned} H_{\rm tun} = & -t_{\rm R}\cos\left[\phi_{\rm R}(x_0) - \phi_{\rm R}(-x_0) + \beta_{\rm R}\right] \\ & - t_{\rm L}\cos\left[m(\phi_{\rm L}(x_0) - \phi_{\rm L}(-x_0)) + \beta_{\rm L}\right]. \end{aligned} \tag{S7}$$

Here $t_{\rm R}$ is the tunnelling amplitude for e/m right-moving excitations, $t_{\rm L}$ is the tunnelling amplitude for left-moving electrons, and $\beta_{\rm R/L}$ are non-universal phases. Higher-order processes such as pair tunnelling can be easily incorporated in what follows, but we neglect these for simplicity.

Our primary interest here is to understand how $H_{\rm tun}$ couples parafermion modes. For concreteness let us consider the setup in Fig. 3(d) where zero-modes α_2 and α_1' reside on opposite sides of the constriction. Due to the pinning of φ and θ near the domain walls in the figure, one can replace

$$\phi_{\rm R/L}(-x_0) = \frac{\pi}{m}\left(\hat{n}_\varphi^{(1)} \pm \hat{n}_\theta^{(2)}\right) \tag{S8}$$

$$\phi_{\rm R/L}(x_0) = \frac{\pi}{m}\left(\hat{n}_\varphi^{(2)} \pm \hat{n}_\theta^{(3)}\right), \tag{S9}$$

yielding

$$\begin{aligned} H_{\rm tun} = & -t_{\rm R}\cos\left[\frac{\pi}{m}\left(\hat{n}_\varphi^{(2)} + \hat{n}_\theta^{(3)} - \hat{n}_\varphi^{(1)} - \hat{n}_\theta^{(2)}\right) + \beta_{\rm R}\right] \\ & - t_{\rm L}\cos\left[\pi\left(-\hat{n}_\varphi^{(2)} + \hat{n}_\theta^{(3)} + \hat{n}_\varphi^{(1)} - \hat{n}_\theta^{(2)}\right) + \beta_{\rm L}\right]. \end{aligned} \tag{S10}$$

The first line simply corresponds to the $(t\alpha_2^\dagger\alpha_1' + H.c.)$ parafermion hybridization term in Eq. (16). Importantly, the second line is a function of the same linear combination of operators $\hat{n}_\varphi^{(2)} + \hat{n}_\theta^{(3)} - \hat{n}_\varphi^{(1)} - \hat{n}_\theta^{(2)}$ that appears in $t_{\rm R}$ above. Consequently electron tunnelling results in a benign coupling of the form $[\tilde{t}(\alpha_2^\dagger\alpha_1')^m + {\rm h.c.}]$ that does not change the conclusions in the Parafermion Braiding section of the main text (apart, perhaps, from an unimportant modification of the non-universal parameter k appearing in the braid transformation). The factor of π in the cosine—which reflects the fact that $t_{\rm L}$ describes tunnelling of electrons rather than charge e/m quasiparticles—underlies this important result. If fractionalized quasiparticles could tunnel between right- and left-movers, then additional non-local terms would appear involving parafermions far from the constriction. For example, tunnelling of left-moving e/m quasiparticles would produce a term of the form $[\delta t(\alpha_2'^2)^\dagger\alpha_1'\alpha_2 + {\rm h.c.}]$. Such couplings, when combined with right-moving e/m tunnelling, would spoil the topological nature of the braids we analyzed but fortunately are precluded in our system.

Controlled-phase gate from sequential braids

In this final section we consider sequential braids that produce a controlled-phase gate. For convenience, we relabel the ground state basis in terms of

$$|q\rangle_k' = |q + k + m\rangle \tag{S11}$$

so that Eq. (20) yields simply

$$U_{12}|q\rangle_k' = {\rm e}^{-{\rm i}\frac{\pi}{2m}q^2}|q\rangle_k'. \tag{S12}$$

Henceforth we drop the label k, assuming that all exchanges occur through a single junction characterized by the same k.

Using this definition, we now examine the effects of more complicated braids. As a concrete example consider Fig. 3(b) where two pairs of domain walls bind four parafermion zero-modes $\alpha_{1,\ldots,4}$, yielding a ground state space with $(2m)^2$ states (assuming no overall fusion channel constraint). For ease of notation let us refer to the domain walls binding α_j simply by j. One can implement a clockwise exchange of the left and right pairs of domain walls via the individual clockwise exchanges of 2 and 3, followed by 3 and 4, 1 and 2, then 2 and 3 once more, building up the unitary $U = U_{23}U_{12}U_{34}U_{23}$.

In order to determine the effects of this braid, we first fix the relative phases of the ground states by defining $|p,q\rangle' = \alpha_1^{\dagger p}\alpha_3^{\dagger q}|0\rangle' \otimes |0\rangle'$, where the first term in the direct product denotes the combined state of parafermions α_1 and α_2, while the second denotes the combined state of parafermions α_3 and α_4. Note that with our conventions we have

$$\alpha_1^\dagger \alpha_2 |0\rangle' = -\mathrm{e}^{\mathrm{i}\frac{\pi}{m}(k+m-1/2)}|0\rangle', \tag{S13}$$

and similarly for $\alpha_3^\dagger \alpha_4$.

The total effect of U is to transform

$$\begin{aligned} \alpha_1 &\to \mathrm{e}^{2\pi\mathrm{i}(k-1)/m}\alpha_3 \\ \alpha_2 &\to \mathrm{e}^{2\pi\mathrm{i}(k-1)/m}\alpha_4 \\ \alpha_3 &\to \alpha_3^{\dagger 2}\alpha_1\alpha_4^2 \\ \alpha_4 &\to \alpha_3^{\dagger 2}\alpha_2\alpha_4^2. \end{aligned} \tag{S14}$$

In particular, $U\alpha_1^\dagger\alpha_2 U^\dagger = \alpha_3^\dagger\alpha_4$ and $U\alpha_3^\dagger\alpha_4 U^\dagger = \alpha_1^\dagger\alpha_2$, which implies that $U|p,q\rangle' = \mathrm{e}^{\mathrm{i}\kappa_{pq}}|q,p\rangle'$. The above relations allow one to determine the phases κ_{pq}; upon discarding an overall phase we obtain

$$U|p,q\rangle' = \mathrm{e}^{\mathrm{i}\frac{\pi}{m}[2k(p-q)-pq]}|q,p\rangle'. \tag{S15}$$

Double exchange of these two pairs of domain walls thus yields, again up to an overall phase,

$$U^2|p,q\rangle' = \mathrm{e}^{-\mathrm{i}\frac{2\pi}{m}pq}|p,q\rangle'. \tag{S16}$$

corresponding to the CP gate from Eq. (21) upon transforming back to our original basis. (Note that the junction parameter k cancels out here.) Equation (S16) constitutes a rather important result: because the phase factor on the right side depends on both p and q, this braid operation can entangle the two registers when acting on a superposition of orthogonal ground states. This braid distinguishes parafermions from Majoranas, since U^2 is trivial when $m = 1$.

The above braids, together with the $(2m)^{\mathcal{N}}$ fold degeneracy of the ground state manifold, suggest the following set of fusion rules for the parafermion modes:

$$\alpha \times \alpha = \psi_0 + \psi_1 + \cdots + \psi_{2m-1} \tag{S17}$$
$$\alpha \times \psi_j = \alpha \tag{S18}$$
$$\psi_{j_1} \times \psi_{j_2} = \psi_{(j_1+j_2) \bmod 2m}. \tag{S19}$$

Here ψ_0 is the identity channel and $\psi_{1,\ldots,2m-1}$ represent the distinct quasiparticle types to which pairs of parafermions can fuse. (The first line can be intuitively understood from our analysis of the parafermionic Josephson effect.) The parafermion α has quantum dimension $\sqrt{2m}$, while each other field has dimension 1. Because the quantum dimension of α squares to an integer, it follows from Ref. 49 that braiding alone does not allow for universal topological quantum computation, consistent with our discussion in the main text. Note that the set of ψ_j form an Abelian sub-algebra consistent with the braid operator U found above. For $m = 1$, these fusion rules reduce to the well-known Ising anyon theory.

Supplementary Bibliography

[49] Rowell, E., Stong, R.& Wang, Z. On classification of modular tensor categories. Comm. Math. Phys. volume 292, pages 343–389 (2009).